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To see a world in a grain of sand

Shou-Cheng Zhang

Department of Physics, Stanford University, Stanford, CA 94305

Throughout John Wheeler's career, he wrestled with big issues like the fundamental length, the black hole and the unification of quantum mechanics and relativity. In this essay, I argue that solid state physics – historically the study of silicon, semiconductors and sand grains – can give surprisingly deep insights into the big questions of the world.

Contents

I. INTRODUCTION

Modern physics is built upon three principal pillars, quantum mechanics, special and general relativity. Historically, these principles were developed as logically independent extensions of classical Newtonian mechanics. While each theory constitutes a logically self-consistent framework, unification of these fundamental principles encountered unprecedented difficulties. Quantum mechanics and special relativity were unified in the middle of the last century, giving birth to relativistic quantum field theory. While tremendously successful in explaining experimental data, ultraviolet infinities in the calculations hint that the theory can not be in its final form. Unification of quantum mechanics with general relativity proves to be a much more difficult task and is still the greatest unsolved problem in theoretical physics.

In view of the difficulties involved with unifying these principles, we can ask a simple but rather bold question: Is it possible that the three principles are not logically independent, but rather there is a hierarchical order in their logical dependence? In particular, we notice that both relativity principles can be formulated as statements of symmetry. When applying non-relativistic quantum mechanics to systems with a large number of degrees of freedom, we sometimes find that symmetries can emerge in the low energy sector, which are not present in the starting Hamiltonian. Therefore, there is a logical possibility that one could start from a single non-relativistic Schrödinger equation for a quantum many-body problem, and discover relativity principles emerging in the low energy sector. If this program can indeed be realized, a grand synthesis of fundamental physics can be achieved. Since non-relativistic quantum mechanics is a finite and logically self-consistent framework, everything derived from it should be finite and logically consistent as well.

The Standard Model in particle physics is described by a relativistic quantum field theory and is experimentally verified below the energy scale of $10^3 GeV$. On the other hand, the Planck energy scale, where quantum gravitational force becomes important, is at $10^{19} GeV$. Therefore, we need to extrapolate 16 orders of magnitude to guess the new physics beyond the standard framework of relativistic quantum field theory. It is quite conceivable that Einstein's principle of relativity is not valid at Planck's energy scale, it could emerge at energies much lower compared to the Planck's energy scale through the magic of renormalization group flow. This situation is analogous to one in condensed matter physics, which deals with phenomena at much lower absolute energy scales. The "basic" laws of condensed matter physics are well-known at the Coulomb energy scale of $1 \sim 10eV$; almost all condensed matter systems can be well described by a non-relativistic Hamiltonian of the electrons and the nuclei(Laughlin and Pines, 2000). However, this model Hamiltonian is rather inadequate to describe the various emergent phenomena, like superconductivity, superfluidity, the quantum Hall effect (QHE) and magnetism, which all occur at much lower energy scales, typically of the order of $1meV$. These systems are best described by "effective quantum field theories", not of the original electrons, but of the quasi-particles and collective excitations. In this lecture, I shall give many examples where these "effective quantum field theories" are relativistic quantum field theories or topological quantum field theories, bearing great resemblance to the Standard Model of elementary particles. The collective behavior of many strongly interacting degrees of freedom is responsible for these striking emergent phenomena. The laws governing the quasi-particles and the collective excitations are very different from the laws governing the original electrons and nuclei(Anderson, 1972). This observation inspires us to construct models of elementary particles by conceptually visualizing them as quasi-particles or collective excitations of a quantum many-body system, whose basic constituents are governed by a simple non-relativistic Hamiltonian. This point of view is best summarized by the following diagram:

$$\begin{array}{ccc}
\text{Planck energy at } 10^{19}GeV & \Leftrightarrow & \text{Coulomb energy at } 10eV \\
\uparrow ? & & \downarrow \\
\text{Standard Model at } 10^{3}GeV & \Leftrightarrow & \text{Superconductivity, QHE, Magnetism etc at 1 meV} \\
\text{Relativistic quantum field theory of elementary particles} & & \text{Effective quantum field theory of quasi-particles}
\end{array}$$

The conceptual similarity between particle physics and condensed matter physics has played a very important role in the history of physics. A crucial ingredient of the Standard Model, the idea of spontaneously broken symmetry and the Higgs mechanism, first originated from the BCS theory of superconductivity. This example vividly shows that the physical vacuum is not empty, but a condensed state of many interacting degrees of freedom. Another fundamental concept is the idea of renormalization group transformation, which was simultaneously developed in the context of particle physics and in the study of critical phenomena. From the theory of renormalization group, we learned that symmetries can emerge at the low energy sector, without being postulated at the microscopic level. Today, as physicists face unprecedented challenges of unifying quantum mechanics with relativity, and tackling the problem of quantum gravity, it is useful to look at these historic successes for inspiration. A new era of close interaction between condensed matter physics and particle physics could shed light on these grand challenges of theoretical physics.

II. EXAMPLES OF EMERGENCE IN CONDENSED MATTER SYSTEMS

In this section, we review some well-known examples in condensed matter physics, where one starts from a quantum many-body system at high energies and arrives at a relativistic or topological field theory of the low energy quasi-particles and elementary excitations. The high energy models often look simple and innocuous, yet the emergent low energy phenomena and their effective field theory description are profound and beautiful.

A. 2+1 dimensional QED from superfluid helium films

Let us first start from the physics of a superfluid film. The mean velocity of the helium atoms are significantly lower compared to the speed of light, therefore, relativistic effects of the atoms can be completely neglected. The basic non-relativistic Hamiltonian for this system of identical bosons can be expressed in the following close form:

$$H = \frac{1}{2m} \sum_n \vec{p}_n^{\,2} + \sum_{n<n'} V(x_n - x_{n'}) \tag{1}$$

where V is the inter-atomic potential, whose form depends on the details of the system. However, for a large class of generic interaction potentials, the system flows towards a universal low energy attractive fix point, namely the superfluid ground state. At typical inter-atomic energy scales of a few eV's, helium atoms are the correct dynamic variables, and the Hamiltonian (1) is the correct model Hamiltonian. However, at the energy scale characteristic of the superfluid transition, which is of the order of $1K \sim 10^{-4}eV$, the correct dynamical variables are sound wave modes and the vortices of the superfluid film. (See fig. (1). for an illustration).

The remarkable thing is that the effective field theory model for these low energy degrees of freedom is exactly the relativistic quantum electrodynamics (QED) in $2+1$ dimensions! This connection was established by the work of Ambegaokar, Halperin, Nelson and Siggia(Ambegaokar *et al.*, 1980) and derived from the point of view of vortex duality(Fisher and Lee, 1989). To see how this works, let us recall that the basic hydrodynamical variables of the superfluid film are the density $\rho(x)$ and the velocity $v_i(x)$ fields, $(i = 1, 2)$, satisfying the equation of continuity

$$\partial_t \rho + \partial_i (\bar{\rho} v_i) = 0 \tag{2}$$

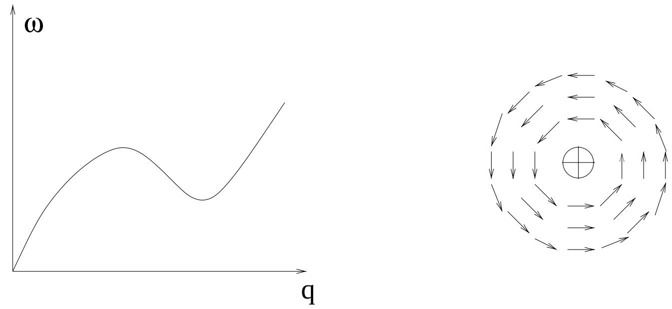

FIG. 1 Collective excitations of a neutral 2D superfluid film are the sound waves and the vortices. In the long wave length limit, the sound wave maps onto the Maxwell fields, while vortices map onto electric charges.

where $\bar{\rho}$ is the average density of the fluid. Now let us recall that in $2+1$ dimensions, the electric field E_i has two components while the magnetic field B has only one component, which can therefore be identified as a scalar. Faraday's law of induction is given by the Maxwell equation:

$$\frac{1}{c}\partial_t B + \epsilon_{ij}\partial_i E_j = 0 \tag{3}$$

where ϵ_{ij} is the antisymmetric tensor in two dimensions. Therefore, if we make the following identification,

$$B \quad \Leftrightarrow \quad -c\frac{\rho}{\bar{\rho}} \qquad\qquad E_i \quad \Leftrightarrow \quad \epsilon_{ij}v_j \tag{4}$$

we see that the equation of continuity of the superfluid film agrees exactly with Faraday's law as expressed in the Maxwell's equation (3). Next we examine the fluid velocity in the presence of a vortex, with unit vorticity, located at the position x_n. The superfluid state has a well defined $U(1)$ order parameter, and the velocity field can be expressed in terms of the phase, ϕ, of the $U(1)$ order parameter:

$$v_i = \frac{\hbar}{m}\partial_i\phi \tag{5}$$

Because of the single valuedness of the quantum mechanical wave function, $e^{i\phi}$ must be single valued. Therefore, the superflow around a vortex is *quantized*:

$$\int \vec{v}\cdot d\vec{l} = 2\pi\frac{\hbar}{m}q \tag{6}$$

where q is an integer. For elementary vortices, $q = \pm 1$. The differential form of this integral equation is

$$\epsilon_{ij}\partial_i v_j = 2\pi\rho_v(x) \tag{7}$$

where $\rho_v(x) = \sum_n q_n\delta(x - x_n)$ is the density of the vortices and $q_n = \pm 1$ is the vorticity. If we identify the vortex density with the electric charge density in Maxwell's equations, we see that equation (7) is nothing but Gauss's law in $2+1$ dimensions:

$$\partial_i E_i = 2\pi\rho_v(x) \tag{8}$$

Next let us investigate the dynamics of the superfluid velocity v_i, through the Josephson equations of superfluidity. The first Josephson equation relates the superfluid velocity to the gradient of the superfluid phase, ϕ, as expressed in (5). The second Josephson equation relates the time derivative of the phase to the chemical potential $\hbar\partial_t\phi = -\mu$. Combining the two Josephson equations, we obtain,

$$\partial_t v_i = \frac{\hbar}{m}\partial_t\partial_i\phi = -\frac{1}{m}\partial_i\mu = -\frac{\kappa}{m\bar{\rho}}\partial_i\rho \tag{9}$$

where we use the compressibility $\kappa = \bar{\rho}\frac{\partial\mu}{\partial\rho}$ to express the chemical potential μ in terms of the density ρ. This equation agrees exactly with the source-free Maxwell equation

$$c\epsilon_{ij}\partial_j B = \partial_t E_i \tag{10}$$

provided one identifies the speed of light as $c^2 = \kappa/m$. This equation needs to be modified in the presence of the vortex flow J_i^v, which unwinds the $U(1)$ phase by 2π each time a vortex passes. The vortex current satisfies the equation of continuity

$$\partial_t \rho_v + \partial_i J_i^v = 0 \tag{11}$$

Therefore, the source free Maxwell equation (10) acquires a additional term, in order to be compatible with both (11) and (8):

$$c\epsilon_{ij}\partial_j B = \partial_t E_i + 2\pi J_i^v \tag{12}$$

This is nothing but Ampere's law, supplemented by Maxwell's displacement current.

This proves the complete equivalence between the superfluid equations and Maxwell's equations in $2+1$ dimensions. Interestingly enough, we seem to have completed a rather curious loop. Starting from the relativistic Standard Model of the quarks and leptons, one arrives at an effective non-relativistic model of the helium atoms (1). However, as one reduces the energy scale further, the effective low energy degrees of freedom become the sound modes and the vortices, which are described by the field theory of $2+1$ dimensional quantum electrodynamics, very similar to the model we started from in the first place! A "civilization" living inside the helium film would first discover the Maxwell's equations, and then, after much harder work, they would establish equation (1) as their "theory of everything".

Superfluid ${}^4He$ films are relatively simple because the ${}^4He$ atom is a boson. The superfluidity of ${}^3He$ is much more complex, with many competing superfluid phases. In fact, Volovik(Volovik, 2001) has pointed out that many phenomena of the superfluid phase of ${}^3He$ share striking similarities with the Standard Model of elementary particles. These similarities inspired him to pioneer a program to address cosmological questions by condensed matter analogs.

B. Dirac fermions of d wave superconductors

Having considered the low energy properties of a superfluid, let us now consider the low energy excitations of a superconductor, with d wave pairing symmetry. In this case, there are low energy fermionic excitations besides the bosonic excitations. This system is realized in the high T_c superconductors. The microscopic Hamiltonian is the two dimensional (2D) Hubbard model, or the $t-J$ model, expressible as

$$H = -t\sum_{\langle ij\rangle,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + J\sum_{\langle ij\rangle} \vec{S}_i\cdot\vec{S}_j \tag{13}$$

where $c_{i\sigma}^\dagger$ is the electron creation operator on site i with spin σ, $\vec{S}_i$ is the electron spin operator and $\langle ij\rangle$ denotes the nearest neighbor bond on a square lattice. Double occupancy of a single lattice site is forbidden.

This model is valid at the energy scale of $150meV$, which is the typical energy scale of the antiferromagnetic exchange J. When the filling factor x lies between 10% and 20%, the ground state of this model is believed to be a d wave superconductor. There is indeed overwhelming experimental evidence that the pairing symmetry of the high T_c superconductor is d wave like. Remarkably, the elementary excitations in this case can be described by the $2+1$ dimensional Dirac Hamiltonian. In contrast to the $t-J$ model, which is valid at the energy scale of $100meV$, the effective Dirac Hamiltonian for the d wave quasi-particles is valid at much lower energy, typically of the order of $30meV$, which is the maximal gap. While the connection between the $t-J$ model and d wave superconductivity still needs to be firmed established, the connection between the d wave BCS quasi-particle Hamiltonian and the Dirac equation is well-known in the condensed matter community(Balents *et al.*, 1998; Franz *et al.*, 2002; Simon and Lee, 1997; Volovik, 1993). Here we follow a pedagogical presentation by Balents, Fisher and Nayak(Balents *et al.*, 1998).

The BCS mean field Hamiltonian for a d wave superconductor is given by

$$H = \sum_{k\alpha}\epsilon_k c_{k\alpha}^\dagger c_{k\alpha} + \sum_k [\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_{k\uparrow}]. \tag{14}$$

where ϵ_k is the quasi-particle dispersion relation, and Δ_k is the d wave pairing gap, given by

$$\epsilon_k = -2t(cosk_x + cosk_y) \ \ , \ \Delta_k = \Delta_0(cosk_x - cosk_y) \tag{15}$$

One can introduce a four component spinor

$$\Upsilon_{a\alpha}(\vec{k}) = \begin{bmatrix} \Upsilon_{11} \\ \Upsilon_{21} \\ \Upsilon_{12} \\ \Upsilon_{22} \end{bmatrix} = \begin{bmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \\ c_{k\downarrow} \\ -c_{-k\uparrow}^\dagger \end{bmatrix}. \tag{16}$$

which doubles the number of degrees of freedom. This can be compensated by summing over only half of the Brillouin zone, say $k_y > 0$. In terms of these variables, the BCS Hamiltonian becomes

$$H = \sum_{k, k_y > 0} \Upsilon^\dagger_{a\alpha}(\vec{k})[\tau^z \epsilon_k + \tau^+ \Delta_k + \tau^- \Delta^*_k]_{ab} \Upsilon_{b\alpha}(\vec{k}), \tag{17}$$

where $\vec{\tau}_{ab}$ are the standard Pauli matrices acting in the particle/hole subspace.

The d-wave nodes are approximately located near the special wave vectors $\vec{K}_1 = (\pi/2, \pi/2)$, $\vec{K}_2 = (-\pi/2, \pi/2)$, $\vec{K}_3 = -\vec{K}_1$ and $\vec{K}_4 = -\vec{K}_2$. In order to obtain a long wave length and low energy description, we can expand around the nodal points $\vec{K}_1$ and $\vec{K}_2$, which satisfy the $k_y > 0$ constraint. The nodal points $\vec{K}_3$ and $\vec{K}_4$ are automatically taken into account in the Υ spinor.

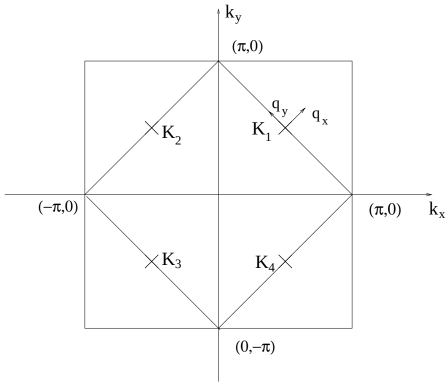

FIG. 2 A 2D d wave superconductor has four nodes, indicated by K_1, K_2, K_3 and K_4. Around these nodal points, BCS quasi-particles obey the massless Dirac equation.

Introducing the rotated coordinates q_x and q_y, as indicated in fig. (2), and the effective spinors

$$\Psi_{1a\alpha}(\vec{q}) = \Upsilon_{a\alpha}(\vec{K}_1 + \vec{q}) \quad , \quad \Psi_{2a\alpha}(\vec{q}) = \Upsilon_{a\alpha}(\vec{K}_2 + \vec{q}) \tag{18}$$

we obtain

$$H = \sum_{q \in K_1} \Psi^\dagger_{1a\alpha}(\vec{q})[\tau^z \epsilon_{K_1+q} + \tau^+ \Delta_{K_1+q} + \tau^- \Delta^*_{K_1+q}]_{ab} \Psi_{1b\alpha}(\vec{q}) + (1 \leftrightarrow 2) \tag{19}$$

Here $q \in K_1$ denotes a momentum sum near the vector K_1. Expansion near K_1 gives

$$\epsilon_{K_1+q} \approx v_F q_x \quad , \quad \Delta_{K_1+q} \approx \Delta q_y \tag{20}$$

A similar expansion applies for K_2. Going to the continuum limit, we obtain the Hamiltonian density

$$\begin{aligned} \mathcal{H} &= \Psi^\dagger_{1a\alpha}[v_F \tau^z i\partial_x + (\tilde{\Delta}\tau^+ + \tilde{\Delta}^* \tau^-) i\partial_y]_{ab} \Psi_{1b\alpha} \\ &+ (1 \leftrightarrow 2; x \leftrightarrow y), \end{aligned} \tag{21}$$

which is exactly the Dirac Hamiltonian density in $2+1$ dimensions. Once again, we see the emergent relativistic behavior of a quantum many-body system. We start from a non-relativistic interacting fermion problem at higher energies, but recover a relativistic Dirac equation at low energies.

C. Emergence of a topological quantum field theory

When Einstein first wrote down his field equation of general relativity, he said that the left-hand side of the equation that had to do specifically with geometry and gravity was beautiful - it was as if made of marble. But the right-hand side of the equation that had to do with matter and how matter produces gravity was ugly - it was as if made of wood. Taking Einstein's aesthetic point of view one step further, one is tempted to construct a fundamental theory by starting with the description of the topology, or a topological field theory without matter and without even geometry from the start. Having demonstrated that the relativistic Maxwell equation and Dirac equation can emerge in the low energy sector of a quantum many-body problem, I now give an example demonstrating how a topological quantum field theory, namely the Chern-Simons (CS) theory, can emerge from the matter degrees of freedom in the low energy sector of the QHE. The CS topological quantum field theory was derived microscopically by Zhang, Hansson and Kivelson(Zhang *et al.*, 1989), and reviewed extensively in ref. (Zhang, 1992).

The basic Hamiltonian of QHE is simply that of a two-dimensional electron gas in a perpendicular magnetic field.

$$H = \frac{1}{2m}\sum_n \left[\vec{p}_n - \frac{e}{c}\vec{A}(x_n)\right]^2 + \sum_n eA_0(x_n) + \sum_{n<n'} V(x_n - x_{n'}) \tag{22}$$

where $\vec{A}$ is the vector potential of the external magnetic field, which in the symmetric gauge can be expressed as

$$A_i = \frac{1}{2}B\epsilon_{ij}x_j \tag{23}$$

A_0 is the scalar potential of the external electric field, $E_i = -\partial_i A_0$, and $V(x)$ is the interaction between the electrons. For high magnetic fields, the electron spins are polarized along the same direction. Since the spin wave function is totally symmetric, the Hamiltonian (22) operates on orbital wave functions which are totally antisymmetric. This model is valid at the Coulombic energy scale of several eV's and has no particular symmetry or topological properties. Since the external magnetic field breaks time reversal symmetry, an invariant tensor ϵ_{ij} can be introduced into the response function, and in particular, one can have a current J_i, which flows transverse to the applied electric field E_j, given by

$$J_i = \rho_H^{-1}\epsilon_{ij}E_j \tag{24}$$

where ρ_H is defined as the Hall resistance. Since the electric field is perpendicular to the induced current, it does no work on the electrons, and the current flow is dissipationless. The 2D electron density n in a magnetic field B is best measured in terms of a dimensionless quantity called the filling factor $\nu = n/n_B$, where $n_B = B/\phi_0 = eB/hc$ is the magnetic flux density. QHE is the remarkable fact that the coefficient of the Hall response is quantized, given by

$$\rho_H = \nu^{-1}\frac{h}{e^2} \tag{25}$$

when the filling fraction is near a rational number $\nu = p/q$ with odd denominator q. QHE at fractional values of ν is referred to as the fractional QHE (FQHE).

FQHE is described by Laughlin's celebrated wave function. There is also an alternative way to understand this profound effect by the Chern-Simons-Landau-Ginzburg (CSLG) effective field theory(Zhang, 1992). The idea is to perform a singular gauge transformation on (22), and turn electrons into bosons. This is only possible in $2+1$ dimensions. Consider another Hamiltonian

$$H' = \frac{1}{2m}\sum_n \left[\vec{p}_n - \frac{e}{c}\vec{A}(x_n) - \frac{e}{c}\,\vec{a}(x_n)\right]^2 + \sum_n eA_0(x_n) + \sum_{n<n'} V(x_n - x_{n'}) \tag{26}$$

Every symbol in H' has the same meaning as in H, except the new vector potential $\vec{a}$, which describes a *gauge interaction* among the particles and is given by

$$\vec{a}(x_n) = \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{n'\neq n}\vec{\nabla}\,\alpha_{nn'} \tag{27}$$

where $\phi_0 = hc/e$ is the unit of flux quantum and $\alpha_{nn'}$ is the angle sustained by the vector connecting particles n and n' with an arbitrary vector specifying a reference direction, say the $\hat{x}$ axis. The crucial difference here is while H operates on a fully antisymmetric fermionic wave function, H' operates on a fully symmetric bosonic wave function. One can prove an exact theorem which states that these two quantum eigenvalue problems are equivalent to each other when $\theta/\pi = (2k+1)$ is an odd integer. In this case, each electron is attached to an odd number of fictitious quanta of gauge flux (cause by a), so that their exchange statistics in $2+1$ dimensions becomes bosonic. These bosons, called composite bosons(Girvin and Macdonald, 1987; Read, 1989; Zhang *et al.*, 1989), see two different types of gauge fields: the external magnetic field A, and an internal statistical gauge field a. The average of the internal statistical gauge field depends on the density of the electrons. When the external magnetic field is such that the filling fraction $\nu = n_B/n = 1/2k+1$ is the inverse of an odd integer, we can always choose $\theta = (2k+1)\pi$ so that the net field seen by the composite bosons cancels each other on the average.

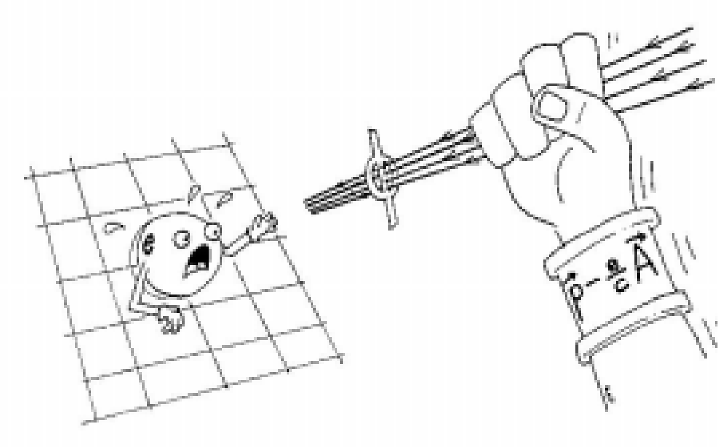

FIG. 3 An electron just before the flux transmutation operation. (taken from the PhD thesis of D. Arovas, illustrated by Dr. Roger Freedman).

The statistical transmutation from electrons to composite bosons can be naturally implemented in quantum field theory through the Cherm-Simons term. The Chern-Simons Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(\frac{\pi}{\theta}\right)\frac{1}{\phi_0}\,\varepsilon^{\mu v \rho}\,a_\mu\,\partial_v\,a_\rho - a_\mu\,j^\mu \tag{28}$$

here j^μ is the current of the composite boson, and $\mu = 0, 1, 2$ is the space-time index in $2+1$ dimensions. The equation of motion for the a_0 field is

$$\epsilon^{ij}\,\partial_i\,a_j(x) = \phi_0\,\frac{\theta}{\pi}\,\rho(x) \tag{29}$$

whose solution for $\rho(x) = \sum_n \delta(x - x_n)$ exactly gives the statistical gauge field in (27).

Now we can present the key argument of the CSLG theory(Zhang, 1992) of QHE. Even though course the statistical transformation can be performed in any $2 + 1$ dimensional systems, this does not mean that the low energy limit of any $2+1$ dimensional system is given by a CS theory, since the partition function also involves the integration over the matter fields j^μ in the second term of (28). The key observation is that at the special filling factors of $\nu = 1/2k+1$, the combined external and statistical magnetic field seen by the composite boson vanishes, therefore, composite bosons naturally condense into a superfluid state. This is the "magic" of the magic filling factors $\nu = 1/2k + 1$. We already showed in section (II.A) that the effective field theory of a $2+1$ dimensional bosonic condensate is the $2+1$ dimensional Maxwell theory. Therefore, the integration over the matter fields in (28) gives the Maxwell Lagrangian, $f^2_{\mu\nu}$. In $2 + 1$ dimensions, the CS term contains one fewer derivative compared with the Maxwell term, it therefore dominates in the long-wave length and low-energy limit. Therefore, the effective Hamiltonian of FQHE is just the topological CS theory, without the matter current term in (28).

Matter degrees of freedom in the starting Hamiltonian (22) are magically turned into topological degrees of freedom of the CS field theory. Alchemy works! Wood is turned into marble! Many people argued that a quantum theory of gravity should be formulated independent of the background metric. The emergent CSLG theory starts from matter degrees of freedom in a background setting, but the resulting effective field theory is independent of the background metric. This demonstrates that in principle, background independent theory can indeed be constructed from non-relativistic quantum many-body systems. In fact, the CSLG theory also leads to a beautiful duality symmetry based on the discrete $SL(2, Z)$ group, very similar to the duality symmetry in the 4D Seiberg-Witten theory. This duality symmetry is again emergent, and it predicts the global phase diagram of the QH Hall system. The phase diagram has a beautiful fractal structure, with one phase nested inside each other, iterated *ad infinitum*(Kivelson *et al.*, 1992).

III. THE FOUR DIMENSIONAL QUANTUM HALL EFFECT

In the previous sections we saw that the collective behavior of quantum many-body systems often gives rise to novel emergent phenomena in the low energy sector, which are described in terms of relativistic or topological quantum field theories. Therefore, one can't help but wonder if the Standard Model could also work this way. The problem is that the well-understood examples of emergent relativistic behaviors in quantum many-body systems work only for lower dimensions, and these models do not have sufficient richness yet. In order for the Stanford Model to appear as emergent behavior, we are led to study higher-dimensional quantum many-body systems, specially the higher-dimensional generalizations of QHE.

A. The model

Of all the novel quantum many-body systems, QHE plays a very special role: it is the only well understood condensed matter system whose low energy limit is a topological quantum field theory. Unlike most other emergent phenomena, like superconductivity and magnetism, QHE works only in two spatial dimensions. There are various ways to see this. First of all, the Hall current is non-dissipative. For the electric field to do no work on the current, the current must flow in a direction perpendicular to the direction of the electric field. In two spatial dimensions, given the direction of the electric field, there is an unique transverse direction for the Hall current, given by equation (24). Since the current and the electric field both carry spatial vector indices, the response must therefore be a rank-two tensor. But there are no natural rank-two antisymmetric tensors in higher dimensions! Secondly, both the single-particle wave function and Laughlin's many-body wave function make extensive use of complex coordinates of particles, which can only be done in two spatial dimensions. This suggests that the higher-dimensional generalization of QHE would necessarily involve a higher-dimensional generalization of complex numbers and analytic functions. In fact, both of these considerations lead to the same higher-dimensional structure, as we shall explain below.

In higher dimensions, given a direction of the electric field, there is no unique transverse direction for the Hall current to flow. However, this statement holds only if we consider the $U(1)$ charge current. If the underlying particles – and the associated currents – carry a non-abelian, *e.g.* $SU(2)$ quantum number, an unique prescription for the current can be given in *four* dimensions. In four dimensions, given a fixed direction of the electric field, say along the x_4 direction, there are three transverse directions. If the current carries a $SU(2)$ isospin label, it also has three internal isospin directions. In this case, the current can flow exactly along the direction in which the isospin is pointing. In this prescription, no preferential direction in space or isospin is picked. The system is invariant under a *combined* rotation of space and isospin. To be more precise, the mathematical generalization of (24) in four dimensions is

$$J^i_\mu = \sigma \eta^i_{\mu\nu} E_\nu \tag{30}$$

Here σ is the generalized Hall conductivity, $\eta^i_{\mu\nu}$ is the t' Hooft tensor, explicitly given by $\eta^i_{\mu\nu} = \epsilon_{i\mu\nu 4} + \delta_{i\mu}\delta_{4\nu} - \delta_{i\nu}\delta_{4\mu}$ and J^i_μ is the isospin current and E_ν is the electric field. Here $\mu, \nu = 1, 2, 3, 4$ label the spatial directions and $i = 1, 2, 3$ label the isospin directions. From (30), we see easily that if E_ν points along the x_4 direction, the current flows along the $x_{1,2,3}$ directions, explicitly determined by the direction of the isospin. Therefore, the t' Hooft tensor is exactly the rank-two antisymmetric tensor we were looking for! The occurrence of the t' Hooft tensor suggests that this problem must have something to do with the $SU(2)$ instanton(Belavin *et al.*, 1975), where the t' Hooft tensor was first introduced. It is not only an *invariant* tensor under combined spatial and isospin rotations, it also satisfies a self-duality condition:

$$\eta^i_{\mu\nu} = \epsilon_{\mu\nu\rho\lambda}\eta^i_{\rho\lambda} \tag{31}$$

Self-duality and anti-self-duality are the hallmarks of the $SU(2)$ Yang-Mills instanton.

Now let us motivate the problem from the point of view of generalizing complex numbers. The natural generalizations of complex numbers are quaternionic numbers, first discovered by Hamilton. A quaternionic number is expressed as $q = q_0 + q_1 i + q_2 j + q_3 k$, where i, j, k are the three imaginary units. This again suggests that the most natural generalization of QHE is from 2D to 4D, where quaternionic numbers can be interpreted as the coordinates of particles in four dimensions. Unlike complex numbers, quaternionic numbers do not commute with each other. In fact, the three imaginary units of quaternionic numbers can be identified with the three generators of the $SU(2)$ group. This suggests that the underlying quantum mechanics problem should involve a non-abelian $SU(2)$ gauge field.

Our last motivation to generalize QHE comes from its geometric structure. As pointed out by Haldane(Haldane, 1983), a nice way to study QHE is by mapping it to the surface of a 2D sphere S^2, with a Dirac mangnetic monopole at its center. (see Fig. 4). The Dirac quantization condition implies that the product of the electric charge, e, and the magnetic charge, g, is quantized, *i.e.* $eg = S$, where $2S$ is a integer. The number $2S + 1$ is the degeneracy of the lowest Landau level. The reason for the existence of a magnetic monopole over S^2 is a coincidence between algebra and geometry. In order for the monopole potential to be topologically non-trivial, the gauge potentials extended from the north pole and the south pole have to match non-trivially at the equator. Since the equator, S^1, and the gauge group, $U(1)$, are isomorphic to each other, a non-trivial winding number exists. Therefore, one may ask whether there are other higher-dimensional spheres for which a similar monopole structure can be defined. This naturally leads to the requirement that the equator of a higher-dimensional sphere to be isomorphic to a mathematical group. This coincidence occurs only for the four sphere, S^4, whose equator, S^3, is isomorphic to the group $SU(2)$. This coincidence between algebra and geometry leads to the first two Hopf maps, $S^3 \to S^2$ and $S^7 \to S^4$.

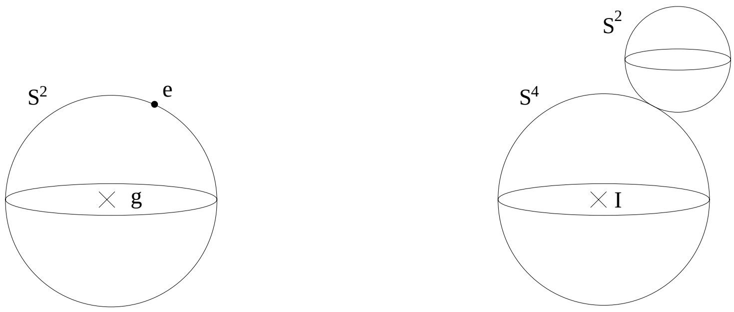

FIG. 4 The 2D QHE consists of electrons e on the surface of a 2D sphere S^2, with a $U(1)$ magnetic monopole g at its center. Similarly, the 4D QHE can be defined on the surface of a 4D sphere S^4, with a $SU(2)$ monopole I at its center. In the large I limit, the $SU(2)$ isospin degree of freedom is S^2.

Therefore, all three considerations – the physical motivation of the transverse current, the mathematical motivation of generalizing complex numbers to quaternionic numbers and the geometric consideration of non-trivial monopole structures – lead to the same conclusion: A non-trivial QHE liquid can be defined in four spatial dimensions (4D) with a $SU(2)$ non-abelian gauge group. Recently, Hu and I (ZH) indeed succeeded in constructing such a model for the 4D QHE(Zhang and Hu, 2001). The microscopic Hamiltonian describes a collection of N fermionic particles moving on S^4, interacting with a $SU(2)$ background isospin gauge potential A_a. It is explicitly defined by

$$H = \frac{\hbar^2}{2MR^2} \sum_{a<b} \Lambda_{ab}^2 \tag{32}$$

where M is the mass of the fermionic particle, R is the radius of S^4, and $\Lambda_{ab} = -i(x_a D_b - x_b D_a)$ is the gauge covariant angular momentum operator. Here x_a is the coordinate of the fermionic particle and $D_a = \partial_a + A_a$ is the gauge invariant momentum operator. The gauge potential A_a $(a = 1, 2, 3, 4, 5)$ is given by

$$A_\mu = \frac{-i}{1 + x_5} \eta^i_{\mu\nu} x_\nu I_i \quad , \quad A_5 = 0 \tag{33}$$

where I_i are the generators of the $SU(2)$ gauge group. An important parameter in this problem is I, the isospin quantum number carried by the fermionic particle. The eigenstates and the eigenvalues of this Hamiltonian can be solved completely, and the spectrum shares many properties with the Landau levels in the 2D QHE problem. In particular, when I becomes large, the ground state of this problem is massively degenerate, with the degenracy scaling like $D \sim I^3$. In order to keep the energy levels finite in the thermodynamical limit, one is required to take the limit $I \to \infty$ as $R \to \infty$, such that

$$R^2/2I = l^2 \tag{34}$$

is finite. l, called the magnetic length, defines the fundamental length scale in this problem. It gives a natural ultraviolet cut-off in this theory, without breaking any rotational symmetries of the underlying Hamiltonian.

While the 4D QH liquid can be elegantly defined on S^4, with the full isometry group as the symmetry of the Hamiltonian, it can also be defined on R^4, with more restricted symmetries. This construction has recently been given by Elvang and Polchinski(Elvang and Polchinski, 2002).

B. Properties of the model

The 2D QH liquid has many interesting properties including incompressibility of the quantum liquid, fractional charge and statistics of elementary excitations, a topological field theory description of the low energy physics, a realization of non-commutative geometry and relativistic chiral excitations at the edge of the QH droplet. Most of these properties also carry over to the QH liquid constructed by ZH. When one completely fills the massively degenerate lowest energy ground states with fermionic particles, with filling factor $\nu \equiv N/D = 1$, one obtains an incompressible quantum liquid, with a finite excitation gap towards all excited states. FQH states can also be constructed for filling fractions $\nu = 1/k^3$, where k is a odd integer. Explicit microscopic wave functions, similar to Laughlin's wave function for the 2D QHE, can be constructed for these incompressible states. The elementary excitations of the FQH states also carry fractional charge $1/k^3$, providing the first direct generalization of fractional charge in a higher-dimensional quantum many-body system.

As discussed in section (II.C), the low energy physics of the 2D QHE can be described by a topological quantum field theory, the CSLG theory. A natural question is whether the QH liquid constructed by ZH can be described by a topological quantum field theory as well. This construction has indeed been accomplished recently, by Bernevig, Chern, Hu, Toumbas and myself(Bernevig *et al.*, 2002). As explained earlier, while the underlying orbital space for our QH liquid is four dimensional, the fermionic particles also carry a large internal isospin degree of freedom I. Since I scales like R^2, the internal space is 2D, which makes the total configuration space a six-dimensional (6D) manifold. Therefore, our QH liquid can either be viewed as a 4D QH liquid with a large internal $SU(2)$ isospin degrees of freedom, or equivalently, as a 6D QH liquid without any internal degree of freedom. The 6D manifold is CP_3, the complex projective space with three complex (and therefore six real) dimensions. This manifold is locally isomorphic to $S^4 \times S^2$. The deep connection between the four sphere S^4 and the complex manifold CP_3 was first introduced to physics through the twistor program of Penrose(Penrose and MacCallum, 1972) and has been exploited extensively in the mathematical literature. Sparling(Sparling, 2002) has recently pointed out the close connection between the twistor theory and the 4D QHE. Our recent work shows that the low energy effective field theory of our QH liquid is given by an abelian CS theory in $6+1$ dimensions

$$S = \nu \int dt d^6 x A \wedge dA \wedge dA \wedge dA \tag{35}$$

where A is an abelian $U(1)$ gauge field over the total configuration space CP_3, and ν is the filling factor. This theory can also be dimensionally reduced to a $SU(2)$ non-abelian CS theory in $4+1$ dimensions, given by

$$S = \frac{4\pi\nu}{3} \int dt d^4 x Tr \left(\mathbf{A} \wedge d\mathbf{A} \wedge d\mathbf{A} - \frac{3i}{2} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge d\mathbf{A} - \frac{3}{5} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right) \tag{36}$$

where $\mathbf{A}$ is a $SU(2)$ matrix-valued gauge field over the orbital space S^4. The precise equivalence of these two models parallels the two equivalent views of our QH liquid mentioned earlier.

An interesting property which arises from this field theory is the concept of duality. As discussed in section (II.C), there is a natural particle-flux duality in the 2D QHE problem: An electron can be represented as a boson with an odd number of flux quanta attached to it. In the new QH liquid, there are other extended objects, namely 2-branes and 4-branes besides the basic fermionic particle, which can be viewed as a 0-brane. Each one of these extended objects is dual to a generalized flux, according to the following table:

Particle	$\Longleftrightarrow$	6-flux
Membrane	$\Longleftrightarrow$	4-flux
4-brane	$\Longleftrightarrow$	2-flux

In the 2D QH problem, the Laughlin quasi-particles obey fractional statistics in 2+1 dimensions. It is natural to ask how fractional statistics generalize in our QH liquid. It turns out that the concept of fractional statistics of point particles can not be generalized to higher dimensions, but fractional statistics for extended objects exist in higher dimensions(Tze and Nam, 1989; Wu and Zee, 1988). In our case, 2-branes have non-trivial statistical interactions which generalizes statistical interactions of Laughlin quasi-particles.

Extended objects like D-branes have been studied extensively in string theory, however, a full quantum theory describing their interactions still needs to be developed. The advantage of our approach is that the underlying microscopic quantum physics is completely specified. Since the extended topological objects emerge naturally from the underlying microscopic physics, there is hope that a full quantum theory can be developed in this case.

The study of 4DQHE is partially motivated by the possibility of emergent relativistic behavior in $3+1$ dimensions. There are several ways to see the connection. First of all, the eigenstates and the eigenfunctions of the Hamiltonian (32) have a natural interpretation in terms of the 4D Euclidean quantum field theory. If we consider a Euclidean quantum field theory as obtained from a Wick rotation of a $3+1$ dimensional compactified Minkowskian quantum field theory, one is naturally lead to consider the eigenvalues and the eigenfunctions of the Euclidean Dirac, Maxwell and Einstein operators on S^4. It turns out that the these eigenvalues and eigenfunctions coincide exactly with the eigenvalues and eigenfunctions of the 4DQHE Hamiltonian (32), where the spins of the relativistic particles are identified with the isospin quantum number, I. The eigenvalue problems of the Dirac, Maxwell and Einstein operators can be directly identified with the Hamiltonian eigenvalue problems for $I = 1/2, 1$ and 2. We mentioned earlier that the underlying fermionic particles constituting our QH liquid have high isospin quantum numbers. However, collective excitations of this QH liquid, which are formed as composite particles, can have low isospin quantum numbers. It is therefore tempting to identify the collective excitations of the QH liquid with the relativistic particles we are familiar with. However, this equivalence is only established in Euclidean space. In order to consider the relationship to Minkowski space, we are naturally lead to the excitations at the boundary, or the edge of our QH liquid.

Let us first review the collective excitations at the edge of a 2D QH liquid. The 2D QH liquid can be confined by a one-body confining potential V. A density excitation is created by removing a particle from the QH liquid and placing it outside of the QH liquid. This way, we have created a particle-hole excitation. If the particle-hole pair moves along a direction parallel to the edge, with a center of mass momentum q_x, the Lorentz force due to the magnetic field acts oppositely on the particle-hole pair, and tries to stretch the pair in the direction perpendicular to the edge. This Lorentz force is balanced by the electrostatic attraction due to the force of the confining potential. Therefore, a unique dipole moment, or a finite separation y of the particle hole pair, is obtained in terms of q_x:

$$y = l^2 q_x \tag{37}$$

On the other hand, the energy of the dipole pair is simply given by $E = V'y$. Here V' is the derivative of the potential evaluated at the edge. Therefore, we obtain a relativistic dispersion relation for the dipole pair

$$E = V'y = l^2 V' q_x \tag{38}$$

with the speed of light given by $c = l^2 V'$. Since the cross product of the gradient of the potential and the magnetic field selects a unique direction along the edge, the excitation is also chiral. In this problem, it can also be shown that not only the dispersion, but also the full interaction is relativistic in the low energy limit. Therefore, the physics at the edge of a 2D QH liquid provides another example of emergent relativistic behavior(Stone, 1990; Wen, 1990).

The physics of the edge excitations of a 2D QH liquid *partially* carries over to our 4D QH liquid(Elvang and Polchinski, 2002; Hu and Zhang, 2002; Zhang and Hu, 2001). Here we can also introduce a confining potential, say around the north pole of S^4, and construct a droplet of the QH fluid. Since our QH liquid is incompressible, the only low energy excitations are the volume preserving shape distortions at the surface. These surface waves can be formed from the particle-hole excitations similar to the ones we described for the 2D QH liquid. A natural speed of light can be introduced, and is given by $c = l^2 V'$. Since our underlying particles carry a large isospin, I, the bosonic composite particle-hole excitations carry all isospins, ranging from 0 to $2I$. The underlying fermionic particles have a strong coupling between their orbital and isospin degrees of freedom. This coupling translates into a relativistic spin-orbital coupling of the bosonic collective excitations. Therefore, excitations with $I = 0, 1, 2$ obey the *free* relativistic Klein-Gordon, Maxwell and Einstein equations. This is an encouraging sign that one might be able to construct an emergent relativistic quantum field theory from the boundary excitations of our 4D QH liquid.

However, there are also many complications which are not yet fully understood in our approach. The most fundamental problem is that particles of our 4D QH liquid carry a large internal isospin, which makes the problem effectively a 6D one. This is the basic reason for the proliferation of higher-spin particles in our theory, an "embarrassment of riches". In addition, there is an incoherent fermionic continuum besides the bosonic collective modes. All these problems can only be addressed when one studies the effects of the interaction carefully. In fact, single particle states in the lowest-Landau-level (LLL) have the full symmetry of $SU(4)$, which is the isometry group of the six dimensional CP_3 manifold. In order to make the problem truly 4D, one needs to introduce interactions which breaks the $SU(4)$ symmetry to a $SO(5)$ symmetry, the isometry group of S^4. This is indeed possible. $SO(5)$ is isomorphic to the group $Sp(4)$. $Sp(4)$ differs from $SU(4)$ by an additional reality condition, implemented through a charge conjugation matrix R. Therefore, any interactions which involve this R matrix would break the symmetry from $SU(4)$ to $SO(5)$, and the geometry of S^4 would emerge naturally. In the strong coupling limit, low energy excitations are not particles but membranes. This reduces the entropy at the edge from $R^3 \times R^2$ to R^3, and is the first step towards solving the problem of "embarrassment of riches".

C. Space, time and the quantum

The 2D QH problem gives a precise mathematical realization of the concept of non-commutative geometry(Douglas and Nekrasov, 2001). In the limit of high magnetic field, we can take the limit of $m \to 0$, so that all higher Landau levels are projected out of the spectrum. In this limit, the equation of motion for a charged particle in an uniform magnetic field B and a scalar potential $V(x, y)$ is given by

$$\dot{x} = l^2 \frac{\partial V}{\partial y} \quad , \quad \dot{y} = -l^2 \frac{\partial V}{\partial x} \tag{39}$$

We notice that the equations for x and y look exactly like the Hamilton equations of motion for p and q. Therefore, this equation of motion can be derived as quantum Heisenberg equations of motion if we postulate a similar commutation relation:

$$[x, y] = il^2 \tag{40}$$

Therefore, the 2D QHE provides a physical realization of the mathematical concept of non-commutative geometry, in which different spatial components do not commute. Early in the development of quantum field theory, this feature has been suggested as a way to cut off the ultraviolet divergences of quantum field theory. In quantum mechanics, the non-commutativity of q and p leads to the Heisenberg uncertainty principle and resolves the classical catastrophe of an electron falling towards the atomic nucleus. Similarly, non-commutativity of space and time could cut off the ultraviolet space-time fluctuations in quantum gravity(Douglas and Nekrasov, 2001). However, the problem is that equation (40) can not be easily generalized to higher dimensions, since one needs to pick some fixed pairs of non-commuting coordinates. Our QH liquid provides a physical realization of non-commutative geometry in four dimensions. The generalization of equation (40) becomes

$$[X_\mu, X_\nu] = 4il^2 \eta^i_{\mu\nu} n_i \tag{41}$$

where X_μ's are the four spatial coordinates and n_i is the isospin coordinate of a particle. This structure of non-commutative geometry is invariant under a combined rotation of space and isospin and treats all these coordinates on equal footing. It is tempting to identify l in equation (41) as the Planck length, which provides the fundamental cutoff of the length scale according to the quantization rule (41). In our theory, however, we know what lies beyond the Planck length: the degrees of freedom are those associated with the higher Landau levels of the Hamiltonian (32).

At this point, it would be useful to discuss the possible implications of (41) on the quantum structure of space-time. In the 4D QH liquid, there is no concept of time. Since all eigenstates in the LLL are degenerate, there is no energy difference which can be used to measure time according to the quantum relation $\Delta t = \hbar/\Delta E$. However, at the boundary of the 4D QH liquid, an energy difference is introduced through the confining potential. The left hand side of equation (41) involves four coordinates. Three of them are the spatial coordinates parallel to the boundary. The fourth coordinate, perpendicular to the boundary, measures the energy difference, and therefore measures time. The commutator among these coordinates implies a quantization procedure. The right hand side of this equation involves the Planck length and the spin. Therefore, this simple equation seems to unify all the fundamental physical concepts: space, time, the quantum, the Planck length and spin in a simple and elegant fashion. It would be nice to use it as a basis to construct a fundamental physical theory.

D. Magic liquids, magic dimensions, magic convergence?

So far our philosophical point of view and our model seem to be drastically different from the approach typical of string theory. However, after the discovery of the new QH liquid, a surprising pattern starts to emerge. Soon after the construction of the new 4D QH liquid, Fabinger(Fabinger, 2002) found that it could be implemented as certain solutions in string theory. Moreover, close examination of this pattern reveals remarkable mathematical similarities not only between these two approaches, but also with other fundamental ideas in algebra, geometry, supersymmetry and the twistor program on quantum space time. The following table summarizes the connections.

Division Algebras:	Real Namubers	Complex Numbers	Quaternions	Octonions
Hopf maps:	$S^1 \to S^1$	$S^3 \to S^2$	$S^7 \to S^4$	$S^{15} \to S^8$
QH liquids:	Luttinger liquid?	Laughlin liquid	ZH liquid	?
Random matrix ensembles:	Orthogonal	Unitary	Symplectic	?
Fractional statistics:	?	particles	membranes	?
Geometric phase:	Z_2	$U(1)$	$SU(2)$	?
Non-commutative geometry:	?	$[X_i, X_j] = il^2\epsilon_{ij}$	$[X_\mu, X_\nu] = 4il^2\eta^i_{\mu\nu}n_i$	?
Twistor transformation:	$SO(2,1) = SL(2,\mathbf{R})$	$SO(3,1) = SL(2,\mathbf{C})$	$SO(5,1) = SL(2,\mathbf{H})$	$SO(9,1) = SL(2,\mathbf{O})$
$N = 1$ SUSY Yang-Mills:	$d = 2+1$	$d = 3+1$	$d = 5+1$	$d = 9+1$
Green-Schwarz Superstring	$d = 2+1$	$d = 3+1$	$d = 5+1$	$d = 9+1$

The construction of the twistor transformation, the $N = 1$ supersymmetric Yang-Mills theory and the Green-Schwarz superstring rely on certain identities of the Dirac Gamma matrices, which work only in certain magic dimensions. In these dimensions, there is an exact equivalence between the Lorentz group and the special linear tranformations of the real, complex, quaternionic and octonic numbers. Our work shows that QH liquids work only in certain magic dimensions exactly related to the division algebras as well! In fact the *transverse* dimensions $((D+1)-2)$ of these relativistic field theories match exactly with the *spatial* dimensions of the quantum liquids. The missing entries in this table strongly suggests that an octonionic version of the QH liquid should exist and may be deeply related to the superstring theory in $d = 9 + 1$. QH liquids exist only in magic dimensions, have membranes and look like a matrix theory. They may be mysteriously related to the M theory after all!

IV. CONCLUSION

Fundamental physics is faced with historically unprecedented challenges. Ever since the time of Galileo, experiments have been the stepping stones in our intellectual quest for the fundamental laws of Nature. With our feet firmly on the ground, there is no summit too high to reach. However, the situation is drastically different in the present day. We are faced with a gap of 16 orders of magnitude between the energy of our experimental capabilities and the summit of Mount Planck. Without experiments, we face the impossible mission of climbing up a waterfall!

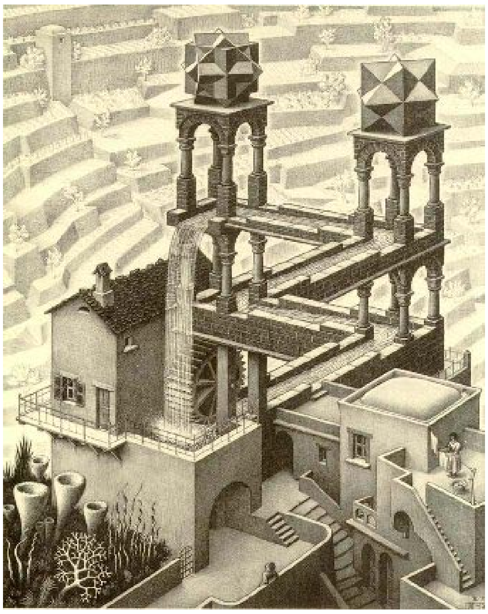

FIG. 5 Esher's waterfall: an alternate passage to Mount Planck?

But maybe there is an alternate passage to Mount Planck. The logical structure of physics may not be a simple one-dimensional line, but rather has a multiply connected or braided topology, very much like Esher's famous *Waterfall*. Instead of going up in energy, we can move down in energy! Atoms, molecules and quantum liquids are made of elementary particles at very high energies. But at low energies, they interact strongly with each other to form quasi-particles, which look very much like the elementary particles themselves! Over the past forty years, we have learned that the strong correlation of these matter degrees of freedom does not lead to ugliness and chaos, but rather to extraordinary beauty and simplicity. The precision of flux quantization, Josephson frequency and quantized Hall conductance are not properties of the basic constituents of matter, but rather are emergent properties of their collective behavior. Therefore, by exploring the connection between elementary particle and condensed matter physics, we can use experiments performed at low energies to understand the physics at high energies. By carrying out the profound implications of these experiments to their necessary logical conclusions, we may learn about the ultimate mysteries of our universe.

Throughout John Wheeler's life, he tackled the big questions of the universe with an unorthodox vision and a poetic flair. Lacking John's eloquence, I simply conclude this tribute to him by reciting William Blake's timeless lines:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

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